

# THE OPEN MAPPING THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

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**Abstract.** This note is devoted to two classical theorems: the open mapping theorem for analytic functions (OMT) and the fundamental theorem of algebra (FTA). We present a new proof of the first theorem, and then derive the second one by a simple topological argument. The proof is elementary in nature and does not use any kind of integration (neither complex nor real). In addition, it is also independent of the fact that the roots of an analytic function are isolated. The proof is based on either the Banach or Brouwer fixed point theorems. In particular, this shows that one can obtain a proof of the FTA (albeit indirect) which is based on the Brouwer fixed point theorem, an aim which was not reached in the past and later the possibility to achieve it was questioned. We close this note with a simple generalization of the FTA. A short review of certain issues related to the OMT and the FTA is also included.

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## 1. INTRODUCTION

The open mapping theorem for analytic functions (OMT) says that any non-locally constant analytic function  $f$  is open (strongly interior), i.e.,  $f(V)$  is open whenever  $V$  is open. The usual proofs of this theorem are based on either Rouché's theorem [7, pp. 306-307], [12, Chapter V], the index (winding number)+argument principle [1, p. 173] or on a simplified version of the minimum modulus principle [11, pp. 256-257]. All these proofs are based on complex integration theory.

Since the theorem “is topological in nature”, mathematicians sought a proof which is topological in nature, and uses minimal analytical theory. In particular, it should not use the “usual sophisticated machinery of analysis” [16], such as development in power series or integration. In various places [5, p. 260], [14, p. 93], [15, p. 2], [17, p. viii] such a proof is regarded as elementary, and this philosophy is well expressed in the book of G. T. Whyburn [17].

In 1952, it seemed that such a proof was found by H. G. Eggleston and H. D. Ursell [5]. This was mentioned by Eggleston and Ursell, and also by Whyburn in his review [16]. The proof is elementary in the sense that one uses only the fact that  $f'$  exists, and does not use the development of  $f$  in power series, or apply integration directly to it. However, a careful reading of the proof shows

that it is based on an integral definition of the index, and also on the properties of the complex exponent function  $t \mapsto \exp(it)$  and the number  $\pi$  (see below), and hence it does use the usual machinery of analysis.

Another attempt to find “an elementary” proof was by C. J. Titus and G. S. Young [14]. They established a theorem which is elementary in the above sense and implies the OMT, assuming one knows that  $f$  is light, i.e., that for any  $c$ , the roots of the equation  $f(z) = c$  are isolated. They however could not establish the lightness of  $f$  (in an elementary way).

Later, by modifications of the arguments in [5], Whyburn [17, p. 76] obtained a proof which bypassed the integral definition of the index and seemed apparently elementary, but again, a careful reading of it shows that it is still based on either integration or power series, since it is based on the definition and properties of the complex exponent function  $t \mapsto \exp(it)$  and the number  $\pi$ . (See [1, pp. 43-46] for discussion and formal definition of them using power series. The definition  $\exp(it) = \cos(t) + i \sin(t)$  which is used in [17, p. 53] can be considered as formal only after one gives a formal non-geometric definition of  $\cos(t)$ ,  $\sin(t)$  and  $\pi$ , and proves some basic trigonometric formulas. See [6, pp. 432-438] where this is done by integrals.)

We note in addition that all these proofs are not elementary in the usual sense, for example because they are long and based on hard theorems, such as the Jordan curve theorem or considerations from degree theory. Especially this is true for that of Whyburn, which is much longer and based on results of several chapters of his book.

Here we present a new proof of the OMT which is elementary in the usual sense of the word (short, not based on hard theorems or hard arguments), and is not based on complex integration theory. In fact, it is not based on integration of any kind. Another property of this proof is that it does not use the fact that  $f$  is light. As far as we know, all known proofs of the OMT, and in particular those mentioned in the references, are based on this fact. Our proof is based on either the Brouwer or the Banach fixed point theorems (so in fact it can be considered as elementary only in the latter case). Since the proof is also based on the theory of power series [1, pp. 39-46], and on the Weierstrass definition of analytic function as one which can be locally developed in power series, it is not elementary in the sense described earlier.

As a corollary of this theorem we obtain the fundamental theorem of algebra (FTA) by a simple topological argument ( $\mathbb{C}$  is connected). In particular, this shows that one can obtain a proof (albeit indirect) of the FTA based on the Brouwer fixed point theorem. This is interesting, because in the past there was an attempt to do it (by B. H. Arnold [3]) but this attempt failed [4] due to a serious mistake. Moreover, about 35 years later it was shown by A. Aleman [2] that it is impossible to prove the FTA by the Brouwer fixed point theorem if one tries to use the methods of [3], i.e., that there is no hope to correct the mistake in [3]. This cast doubt on the possibility of proving the FTA by applying the Brouwer fixed point theorem.

It is interesting to note in this connection that there does exist a proof of the FTA which is based on a fixed point theorem (the Lefschetz fixed point theorem; see [8]). In addition, a careful reading of the proof of the FTA given in [18] shows that one of its ingredients is an argument similar to the one appearing in the proof of the Banach fixed point theorem.

We also note that the connection between the OMT and the FTA is not new. For example, R. L. Thompson [13] proved that the FTA is equivalent to the open mapping theorem for polynomials. In addition, the elementary proof of the FTA by S. Wolfenstein [18], and the proof of the generalization of the FTA by M. Reichaw (Reichbach) [10, p. 160] are also related to connectedness and open mappings, but their arguments are different from ours. (They use the fact that  $f$  is locally open when  $f'(x) \neq 0$ , that  $f'(x) = 0$  only on a finite set of points  $A$  when  $f$  is a polynomial, and that  $\mathbb{C} \setminus A$  is connected. In comparison, in our proof we merely use the fact that  $\mathbb{C}$  is connected and it is irrelevant whether  $f'(x)$  vanishes.)

Finally, we note that we were informed by Simeon Reich that the topological argument we use for proving the FTA was also independently mentioned by him in [9], as a remark on Thompson's paper.

## 2. PROOF OF THE OMT AND THE FTA

We need the following simple lemma. It can be easily proved (and improved) by use of integrals ( $T(y) - T(x) = \int_{[x,y]} T'(z)dz$ ), but also without integrals as below.

**Lemma 2.1.** *Let  $B \subseteq \mathbb{C}$  be nonempty and convex. If  $T : B \rightarrow \mathbb{C}$  is differentiable and  $\sup_{\xi \in B} |T'(\xi)| \leq a$ , then  $T$  is Lipschitz on  $B$  with a Lipschitz constant not greater than  $\sqrt{2}a$ .*

*Proof.* We can write  $T = u + iv$  where  $u, v : B \rightarrow \mathbb{R}$  are differentiable. Let  $\xi_1, \xi_2 \in B$  and let  $\gamma(t) = \xi_1 + (\xi_2 - \xi_1)t$ ,  $t \in [0, 1]$  be the line segment connecting them. Since  $|u_x|^2 + |u_y|^2 = |T'|^2 \leq a^2$  by the Cauchy-Riemann equations, the real version of Lagrange's mean value theorem and the Cauchy-Schwarz inequality imply that

$$|u(\xi_1) - u(\xi_2)| = |u(\gamma(0)) - u(\gamma(1))| = |(u(\gamma))'(t)| \leq \|\nabla u(\gamma)\| |\xi_1 - \xi_2| \leq a |\xi_1 - \xi_2|.$$

The same holds for  $v$ . Hence

$$|T(\xi_1) - T(\xi_2)| = \sqrt{|u(\xi_1) - u(\xi_2)|^2 + |v(\xi_1) - v(\xi_2)|^2} \leq \sqrt{2}a |\xi_1 - \xi_2|.$$

□

**Theorem 2.2.** *Let  $f(z)$  be a non-locally-constant analytic function defined in an open set  $V$ . Then  $f$  is open, i.e.,  $f(U)$  is open whenever  $U \subseteq V$  is open.*

*Proof.* Suppose  $U \subseteq V$  is open, and let  $z_0 \in U$  and  $w_0 = f(z_0) \in f(U)$ . It should be proved that there exists  $r > 0$  such that the open ball  $B(w_0, r)$  of radius  $r$  and center  $w_0$  is contained in  $f(U)$ , i.e., that for any  $w \in B(w_0, r)$ , the equation  $w = f(z)$  has a solution  $z \in U$ .

Since  $f$  is analytic, it can be represented as  $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$  in a neighborhood of  $z_0$ . Let  $1 \leq k \in \mathbb{N}$  be the minimal index for which the coefficient  $a_k$  does not vanish. Such  $k$  exists, because otherwise  $f$  is locally constant in that neighborhood of  $z_0$  (and in fact globally constant if  $V$  is also connected, by the identity/uniqueness theorem). Consequently, we can write  $f(z) = a_0 + (z - z_0)^k(a_k + h(z - z_0))$ , where  $a_0 = w_0$  and

$$h(\xi) = \sum_{p=k+1}^{\infty} a_p \xi^{p-k}.$$

By the change of variables  $\xi = z - z_0$ , the equation  $f(z) = w$  becomes

$$(1) \quad \xi^k = \frac{w - w_0}{a_k + h(\xi)}.$$

Now it is tempting to take root and transform this equation to a fixed point equation and then to use a corresponding fixed point theorem, but one should be careful, because there are several candidates for the root, and each one of them has a line of discontinuity. Let  $g_1, g_2$  be the complex functions defined by

$$g_1(\xi) = |\xi|^{\frac{1}{k}} \cdot e^{i \frac{\arg(\xi)}{k}}, \quad g_2(\xi) = |\xi|^{\frac{1}{k}} \cdot e^{i \frac{\text{Arg}(\xi)}{k}}.$$

Roughly speaking,  $g_1$  and  $g_2$  are “ $\xi^{1/k}$ ”. Formally,  $g_1$  and  $g_2$  are right inverses of the function  $G(\xi) = \xi^k$ , i.e.,  $G(g_1(\xi)) = G(g_2(\xi)) = \xi$ . The reversed equalities  $g_1(G(\xi)) = \xi$ ,  $g_2(G(\xi)) = \xi$  are not necessarily true (take  $k = 2$  and  $\xi = -1$  for example). Because  $0 \leq \arg(\xi) < 2\pi$  and  $-\pi \leq \text{Arg}(\xi) < \pi$ ,  $g_1$  and  $g_2$  are continuously differentiable in  $\mathbb{C} \setminus ([0, \infty) \times \{0\})$  and  $\mathbb{C} \setminus ((-\infty, 0] \times \{0\})$  respectively. In other words,  $g_1$  ( $g_2$ ) is continuously differentiable at any  $\xi \neq 0$  with  $\text{Arg}(\xi) \neq 0$  ( $\text{Arg}(\xi) \neq -\pi$ ).

Since  $\lim_{\xi \rightarrow 0} h(\xi) = 0$ , there exists  $0 < \rho$  sufficiently small such that  $B[z_0, \rho] \subset U$ , and such that  $|a_k/2| < |h(\xi) + a_k|$  and  $\text{Arg}(a_k/(h(\xi) + a_k)) \in (-\pi/8, \pi/8)$  for all  $\xi$  in the small closed ball  $B[0, \rho] = \overline{B(0, \rho)}$ .

Let

$$0 < r \leq \min(|a_k/2|\rho^k, (1/(2\alpha))^k),$$

where

$$\alpha = (1/k) \cdot |2/a_k|^{3-\frac{1}{k}} \cdot (1 + \sup_{|\xi| \leq \rho} |h'(\xi)|).$$

The reason for choosing these values will become clear in a moment. We claim that the ball  $B(w_0, r)$  is contained in  $f(U)$ . To see this, fix  $w \in B(w_0, r)$ . It can be assumed that  $w \neq w_0$ , because otherwise  $w = f(z_0)$  and we are done. It suffices to show that (1) has a solution  $\xi \in B[0, \rho]$ . Consider the equation

$$(2) \quad \xi = g \left( \frac{w - w_0}{h(\xi) + a_k} \right) \equiv T(\xi).$$

Here we take  $g = g_1$  if  $\text{Re}((w - w_0)/a_k) < 0$  and  $g = g_2$  otherwise. By applying the function  $G(\xi) = \xi^k$  to (2), we see that any solution of (2) is a solution of (1) (the converse is not necessarily true because it may happen that  $g(G(\xi)) \neq \xi$ ), so it suffices to show that (2) has a solution  $\xi \in B[0, \rho]$ .

$T$  is well defined in  $B[0, \rho]$  by the choice of  $\rho$ . In addition, it is continuously differentiable there, because  $(w - w_0)/(h(\xi) + a_k)$  is outside the discontinuous set (ray) of  $g$ . (For example, if  $g = g_1$ , then  $\text{Arg}((w - w_0)/(h(\xi) + a_k)) = \text{Arg}(a_k/(h(\xi) + a_k)) + \text{Arg}((w - w_0)/a_k) \notin (-\pi/8, \pi/8)$  by the choice of  $\rho$ .)

By the choice of  $\rho$  it follows that  $|T(\xi)| \leq |2r/a_k|^{\frac{1}{k}}$  for  $\xi \in B[0, \rho]$ , so  $T(B[0, \rho]) \subseteq B[0, \rho]$  by the choice of  $r$ . Since  $T$  is continuous, we can finish the proof by applying the Brouwer fixed point theorem to (2). However, we will show below that the more elementary Banach fixed point theorem suffices for this purpose. By the choice of  $r$  and  $\alpha$ ,

$$|T'(\xi)| = \frac{|w - w_0|^{\frac{1}{k}} \cdot |h'(\xi)|}{k \cdot |h(\xi) + a_k|^{3 - \frac{1}{k}}} \leq \alpha \cdot r^{\frac{1}{k}}, \quad \forall \xi \in B[0, \rho],$$

so  $\sup_{\xi \in B[0, \rho]} |T'(\xi)| \leq 0.5$ , again by the choice of  $r$ . Thus, by Lemma 2.1,  $T$  is Lipschitz on  $B[0, \rho]$  with a Lipschitz constant not greater than  $0.5\sqrt{2} < 1$ . Since  $B[0, \rho]$  is a complete metric space, the Banach fixed point theorem implies that (2) has a (unique) solution  $\xi \in B[0, \rho]$ .  $\square$

**Theorem 2.3.** *Let  $1 \leq m \in \mathbb{N}$  and let  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(z) = a_m z^m + \dots + a_1 z + a_0$ , be a polynomial of degree  $m$  ( $a_m \neq 0$ ). Then  $f$  is surjective, i.e.,  $f(\mathbb{C}) = \mathbb{C}$ . In particular,  $0 \in f(\mathbb{C})$ , i.e.,  $f$  has a root.*

*Proof.* Since  $f(\mathbb{C})$  is nonempty ( $f(0) \in f(\mathbb{C})$ ) and open (by Theorem 2.2), it suffices to show that it is closed, because then, the fact that  $\mathbb{C}$  is connected will imply that  $f(\mathbb{C}) = \mathbb{C}$ .

Let  $(z_n)_n$  be a sequence of complex numbers with the property that  $f(z_n) \rightarrow w \in \mathbb{C}$ . In particular,  $(f(z_n))_n$  is bounded. Hence the sequence  $(z_n)_n$  is bounded, because otherwise

$$\overline{\lim}_{n \rightarrow \infty} |f(z_n)| = \overline{\lim}_{n \rightarrow \infty} |(z_n)^m| \cdot |a_m + \frac{a_{m-1}}{z_n} + \dots + \frac{a_0}{(z_n)^m}| = \infty \cdot |a_m| = \infty.$$

Thus  $(z_n)_n$  has a convergent subsequence  $z_{n_k} \rightarrow z \in \mathbb{C}$ , so  $w = f(z)$  since  $f$  is continuous.  $\square$

We finish by remarking that the above proof implies that Theorem 2.3 can be obviously generalized as follows: an entire analytic function  $f$  is surjective if and only if its image is closed. It would be interesting to obtain a simple necessary and/or sufficient condition for this in terms of the coefficients of  $f$ . Unfortunately, with the exception of the condition that almost all its coefficients vanish, i.e., that it is a polynomial, we do not know any such condition.

Nevertheless, one can indeed obtain some simple sufficient conditions for  $f$  to be surjective in terms of a possible representation that it might have. A trivial example is when  $f$  is a composition of two surjective mappings. Another example is when  $f(z) = p(g(z)) + q(1/g(z))$ , where  $p$  and  $q$  are non-constant polynomials and  $g$  is a non-constant analytic function which does not vanish, such as  $g(z) = \exp(\alpha z)$ ,  $\alpha \neq 0$ . Indeed, suppose  $p(z) = \sum_{k=0}^n a_k z^k$ ,  $q(z) = \sum_{k=0}^m b_k z^k$  where  $a_n \neq 0, b_m \neq 0$ , and let  $w \in \mathbb{C}$  be given. Since  $a_n \neq 0, b_m \neq 0$ , the FTA and a simple manipulation imply that there exists  $t \neq 0$  such that

$p(t) + q(1/t) = w$ . Because  $g(\mathbb{C}) = \mathbb{C} \setminus \{0\}$  by Picard's theorem, there exists  $z \in \mathbb{C}$  such that  $g(z) = t$ , so  $f(z) = w$  and  $f$  is surjective. Hence, for instance, the function  $f(z) = \cos^7(z^2 + z + i) - 3 \cos(z^2 + z + i) + 1$  is surjective, and in particular it has a root.

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